Growing-and-decaying mode solution to the Davey-Stewartson equation

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The growing-and-decaying mode solution to the Davey-Stewartson equation are presented, which describe the long time evolution of the Benjamin-Feir unstable mode in two dimensions. A solution consisting of a line soliton and a growing-and-decaying mode shows that the Benjamin-Feir unstable mode does not destroy the structure of the line soliton. The breather solution and rational growing-and-decaying mode solution are also presented. [S1063-651X(99)00708-4]

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I. INTRODUCTION

Benjamin and Feir demonstrated theoretically and experimentally that a uniform train of weakly nonlinear deep water waves is unstable to infinitesimal modulational perturbations [1,2]. The governing equations for the description of the long time evolution of the unstable wave train have been studied by many authors [3–9]. Zakharov [3], Hasimoto and Ono [5], and Davey [6] derived the nonlinear Schrödinger (NLS) equation in the one-dimensional propagation

$$iu_t + pu_{xx} + r|u|^2u = 0. \tag{1}$$

It is well known that the solution is stable to relatively small disturbances only if $pr<0$, which leads to the same stability condition as that found by Benjamin and Feir. The extension to the two-dimensional case was examined by Zakharov [3], Benny and Roskes [8], and Devey and Stewartson [9]. The evolution of a two-dimensional wave packet is described by the Davey-Stewartson (DS) equation. Davey and Stewartson obtained the stability condition for the two-dimensional (2D) propagation by using the reduction of the DS equation to the 1D-NLS equation and the stability condition of 1D-NLS equation [9]. The time evolution of the solution of the 1D-NLS equation with periodic boundary condition and with a Benjamin-Feir unstable initial condition was studied numerically by Lake *et al.* [10]. They found that a modulated unstable wave train achieves a state of maximum modulation and returns to an unmodulated initial state, which is well known as the Fermi-Pasta-Uram (FPU) recurrence. Yuen and Ferguson investigated numerically the long time evolution of the solution of the 2D-NLS equation [11]

$$iu_t + pu_{xx} + qu_{yy} + r|u|^2u = 0. \tag{2}$$

The FPU recurrence was also observed by them for a wide variety of initial conditions. One of the important features of the solutions of the NLS equation in one and two dimensions is the reverting of the unstable wave train to its initial state. The exact solution to describe the nonlinear evolution of the linearly unstable mode was obtained in the Boussinesq equation by Yajima [12]. The solution grows exponentially according to the linear instability at the initial stage $u \simeq e^{\gamma t}\cos(kx)$, where γ is the linear growth rate for the wave number k which reaches the maximum amplitude after some finite time and damps to die out at a sufficiently large time as $u \simeq u_0' e^{-\gamma t}\cos(kx)$. Such a growing-and-decaying mode solution has also been obtained in the sine-Gordon and 1D-NLS equations [13, 14]. Recently, the intermittent wave mode solution by which the growing-and-decaying process is repeated in time was obtained in the Boussinesq equation [15]. They are all recurrent solutions in the one-dimensional propagation. However, we conjecture from the numerical results of the long time evolution of solution to the 2D-NLS equation that exact recurrent solutions can be also obtained in two-dimensional equations.

In this paper, we investigate the recurrent solutions to the DS equation. The purposes of this study are to show that (i) the DS equation has the growing-and-decaying mode solution, breather solution, and rational growing-and-decaying mode solution and (ii) the growing-and-decaying mode solution describes the long time evolution of the Benjamin-Feir unstable modes in two dimensions. The stability of the soliton was examined by Zakharov [16] using the inverse scattering transformation method. He obtained solutions which describe the nonlinear stage of instability and the nonlinear oscillations of the soliton in the stable case. The stability of the line soliton due to nonlocal disturbances is also investigated by using the solution consisting of a line soliton and a growing-and-decaying mode. The solution shows that the linearly unstable mode does not destroy the structure of the line soliton in the same way as the one-dimensional propagation described by the 1D-NLS equation [14].

II. GROWING AND DECAYING MODES

The Davey-Stewartson equation may be written as

$$\begin{gathered} iu_t + pu_{xx} + u_{yy} + r|u|^2u - 2uv = 0, \\ pv_{xx} - v_{yy} - pr(|u|^2)_{xx} = 0, \end{gathered} \tag{3}$$

where $p=\pm 1$, r is constant. Equation (3) with $p=1$ and $p=-1$ are called the DS I and DS II equations, respectively. First we investigate the stability of the plane wave of Eq. (3),

$$\begin{gathered} u^0 = u_0 \exp\{i(kx+ly-\omega t)\}, \\ v^0 = 0, \end{gathered} \tag{4}$$

with the dispersion relation $\omega = pk^2 + l^2 - ru_0^2$, by considering infinitesimal modulational perturbations of the form

$$u=u_0e^{i(kx+ly-\omega t)}[1+\hat{\epsilon}_+(t)\exp\{i(\beta x'+\delta y')\}+\hat{\epsilon}_-(t)\exp\{-i(\beta x'+\delta y')\}],$$

$$v=\hat{\kappa}_+(t)\exp\{i(\beta x'+\delta y')\}+\hat{\kappa}_-(t)\exp\{-i(\beta x'+\delta y')\}, \tag{5}$$

with $\hat{\epsilon}_\pm(t)=\hat{\varepsilon}_\pm(0)\exp(\sigma t)$ and $\hat{\kappa}_\pm(t)=\hat{\kappa}_\pm(0)\exp(\sigma t)$, where

$$x'=x-\frac{\partial\omega}{\partial k}t=x-2pkt,$$

$$y'=y-\frac{\partial\omega}{\partial l}t=x-2lt. \tag{6}$$

Substituting Eq. (5) into Eq. (3), we have

$$\frac{\hat{\varepsilon}_-^*(0)}{\hat{\varepsilon}_+(0)}=\frac{\delta^2+p\beta^2-i\sigma}{\delta^2+p\beta^2+i\sigma}, \tag{7}$$

$$\hat{\kappa}_+(0)=\hat{\kappa}_-^*(0)=-\frac{pru_0^2\beta^2}{\delta^2-p\beta^2}\{\hat{\varepsilon}_+(0)+\hat{\varepsilon}_-^*(0)\}, \tag{8}$$

and

$$\sigma=\pm|\delta^2+p\beta^2|\sqrt{\frac{2ru_0^2}{\delta^2-p\beta^2}-1}. \tag{9}$$

The condition in which the plane wave (4) is unstable to modulational perturbation is given by (i) in the case of $p=1$,

$$2ru_0^2>\delta^2-\beta^2>0 \quad \text{for} \quad r>0,$$

$$2|r|u_0^2>\beta^2-\delta^2>0 \quad \text{for} \quad r<0, \tag{10}$$

and (ii) in the case of $p=-1$,

$$2ru_0^2>\delta^2+\beta^2 \quad \text{for} \quad r>0. \tag{11}$$

Taking $\sigma=|\delta^2+p\beta^2|\sqrt{(2ru_0^2)/(\delta^2-p\beta^2)-1}>0$ and introducing a new parameter $\tilde{\phi}$,

$$\tilde{\phi}=2\sin^{-1}\left(\sqrt{\frac{\delta^2-p\beta^2}{2ru_0^2}}\right)>0, \tag{12}$$

we have

$$\hat{\varepsilon}_-(0)=\begin{cases}-\hat{\varepsilon}_+^*(0)e^{-i\tilde{\phi}} & \text{for} \quad \delta^2+p\beta^2>0,\\ -\hat{\varepsilon}_+^*(0)e^{i\tilde{\phi}} & \text{for} \quad \delta^2+p\beta^2<0,\end{cases} \tag{13}$$

where $0<\tilde{\phi}<\pi$ has been assumed. Then, u and v are expressed by

$$u=\begin{cases}u_0e^{i(kx+ly-\omega t)}[1+\tilde{\epsilon}(1-e^{-i\tilde{\phi}})e^{\sigma t}\cos(\beta x'+\delta y'+\tilde{\theta})] & \text{for} \quad \delta^2+p\beta^2>0, \quad (14)\\ u_0e^{i(ix+ly-\omega t)}[1+\tilde{\epsilon}(1-e^{i\tilde{\phi}})e^{\sigma t}\cos(\beta x'+\delta y'+\tilde{\theta})] & \text{for} \quad \delta^2+p\beta^2<0, \quad (15)\end{cases}$$

where $\tilde{\epsilon}$ is a small real number and $\tilde{\theta}$ is an arbitrary real number. If we take $\sigma=-|\delta^2+p\beta^2|\sqrt{(2ru_0^2)/(\delta^2-p\beta^2)-1}<0$,

$$\hat{\varepsilon}_-(0)=\begin{cases}-\hat{\varepsilon}_+^*(0)e^{i\tilde{\phi}} & \text{for} \quad \delta^2+p\beta^2>0,\\ -\hat{\varepsilon}_+^*(0)e^{-i\tilde{\phi}} & \text{for} \quad \delta^2+p\beta^2<0,\end{cases} \tag{16}$$

and

$$u=\begin{cases}u_0e^{i(kx+ly-\omega t)}[1+\tilde{\epsilon}'(1-e^{i\tilde{\phi}})e^{\sigma t}\cos(\beta x'+\delta y'+\tilde{\theta})] & \text{for} \quad \delta^2+p\beta^2>0, \quad (17)\\ u_0e^{i(kx+ly-\omega t)}[1+\tilde{\epsilon}'(1-e^{-i\tilde{\phi}})e^{\sigma t}\cos(\beta x'+\delta y'+\tilde{\theta})] & \text{for} \quad \delta^2+p\beta^2<0, \quad (18)\end{cases}$$

which is a damping mode solution.

Here, it should be noted that the DS equation (3) is reduced to the 1D-NLS equation when we assume u to be independent of x and when we put $v=0$ (and $y\rightarrow x$). Under the assumption, Eq. (9) becomes $\sigma^2=\delta^2(2ru_0^2-\delta^2)$, which is in agreement with the well known growth rate of the Benjamin-Feir instability in one-dimensional propagation. Therefore, we can consider σ given by Eq. (9) with a plus sign as the growth rate of the Benjamin-Feir instability in the two-dimensional case. It is very interesting to note that the solutions described by the nonlinear evolution of unstable modes can be obtained from the N-soliton solution of Satsuma and Ablowitz [17]. We shall show that such a solution can be constructed from the two-soliton solution with pure imaginary wave numbers and complex frequencies.

The two-soliton solution may be written as [17]

$$u=u_0e^{i(kx+ly-\omega t)}\frac{g}{f},$$

$$v=-2p(\log f)_{xx}, \tag{19}$$

with

$$f=1+e^{\eta_1}+e^{\eta_2}+De^{\eta_1+\eta_2},$$

$$g=1+e^{\eta_1+i\phi_1}+e^{\eta_2+i\phi_2}+De^{\eta_1+\eta_2+i(\phi_1+\phi_2)}, \tag{20}$$

where

$$\omega=pk^2+l^2-ru_0^2, \tag{21}$$

$$\eta_j=K_jx+L_jy-\Omega_jt+\eta_j^0, \tag{22}$$

$$\sin^2\frac{\phi_j}{2}=\frac{pK_j^2-L_j^2}{2ru_0^2}, \tag{23}$$

$$\Omega_j=2pkK_j+2lL_j-(pK_j^2+L_j^2)\cot\frac{\phi_j}{2}\quad (j=1,2). \tag{24}$$

Taking wave numbers and frequencies pure imaginary and complex, respectively, as follows:

$$K_1=i\beta,\quad L_1=i\delta,\quad \Omega_1=\Omega+i\gamma,$$

$$K_2=-i\beta,\quad L_2=-i\delta,\quad \Omega_2=\Omega-i\gamma,$$

$$\phi_1=\phi_2=\phi:\text{real},$$

$$\eta_1^0=\eta_2^{0*},\quad e^{\eta_1^0}=e^{\eta_2^{0*}}=-\frac{1}{2}e^{-\bar{\sigma}+i\theta}, \tag{25}$$

we have the following dispersion relation and D [in Eq. (20)]

$$\sin^2\frac{\phi}{2}=\frac{\delta^2-p\beta^2}{2ru_0^2}>0, \tag{26}$$

$$\Omega=(\delta^2+p\beta^2)\cot\frac{\phi}{2}, \tag{27}$$

$$\gamma=2pk\beta+2l\delta, \tag{28}$$

$$D=\frac{2}{1+\cos\phi}>1. \tag{29}$$

Then, the solution is given by

$$\begin{aligned}u=u_0e^{i(kx+ly-\omega t+\phi)}&[\sqrt{D}\cosh(\Omega t+\sigma-i\phi)\\&-\cos(\beta x+\delta y-\gamma t+\theta)][\sqrt{D}\cosh(\Omega t+\sigma)\\&-\cos(\beta x+\delta y-\gamma t+\theta)]^{-1},\end{aligned}$$

$$v=-2p\beta^2\frac{\sqrt{D}\cosh(\Omega t+\sigma)\cos(\beta x+\delta y-\gamma t+\theta)-1}{[\sqrt{D}\cosh(\Omega t+\sigma)-\cos(\beta x+\delta y-\gamma t+\theta)]^2}, \tag{30}$$

where $\sigma=\tilde{\sigma}+\log(2/\sqrt{D})$. The existence condition for the nonsingular solution (30) is given by $D>1$, which is satisfied for $0<(\delta^2-p\beta^2)/(2ru_0^2)<1$ (for real ϕ) which is in agreement with the conditions (10) and (11) in which the plane wave (4) is unstable to modulational perturbation. This solution grows exponentially at initial stage ($t=-T+t', T\gg 1$) as follows:

$$u=u_0e^{i(kx+ly-\omega t+2\phi)}[1+\epsilon(1-e^{-i\phi})e^{\Omega t'}\cos(\beta x+\delta y-\gamma t+\theta)],$$

$$v=-2p\beta^2\epsilon e^{\Omega t'}\cos(\beta x+\delta y-\gamma t+\theta)\quad\text{for}\quad\Omega>0 \tag{31}$$

or

$$u=u_0e^{i(kx+ly-\omega t)}[1+\varepsilon(1-e^{i\phi})e^{-\Omega t'}\cos(\beta x+\delta y-\gamma t+\theta)],$$

$$v=-2p\beta^2\varepsilon e^{-\Omega t'}\cos(\beta x+\delta y-\gamma t+\theta)\quad\text{for}\quad\Omega<0, \tag{32}$$

where $t'=t+T$, $\epsilon=(2/\sqrt{D})e^{-\Omega T+\sigma}\ll 1$, and $\varepsilon=(2/\sqrt{D})e^{\Omega T-\sigma}\ll 1$, achieves a state of maximum modulation and finally ($t=T'\gg 1$) returns exponentially to the initial state as follows:

$$u=u_0e^{i(kx+ly-\omega t)}[1+\epsilon'(1-e^{i\phi})e^{-\Omega t''}\cos(\beta x+\delta y-\gamma t+\theta)],$$
$$v=-2p\beta^2\epsilon' e^{-\Omega t''}\cos(\beta x+\delta y-\gamma t+\theta)\quad\text{for}\quad \Omega>0 \tag{33}$$

or

$$u=u_0e^{i(kx+ly-\omega t+2\phi)}[1+\varepsilon'(1-e^{-i\phi})e^{\Omega t''}\cos(\beta x+\delta y-\gamma t+\theta)],$$
$$v=-2p\beta^2\varepsilon' e^{\Omega t''}\cos(\beta x+\delta y-\gamma t+\theta)\quad\text{for}\quad \Omega<0, \tag{34}$$

where $t''=t-T'$, $\epsilon'=(2/\sqrt{D})e^{-\Omega T'-\sigma}$, and $\varepsilon'=(2/\sqrt{D})e^{\Omega T'+\sigma}$. Typical time evolutions of this solution are shown in Figs. 1 and 2. These figures show that the solution reaches a state of maximum modulation and after reaching maximum modulation, demodulates and finally returns to an unmodulated state. We call this solution the growing-and-decaying mode solution hereafter. It is interesting to note that the growth rate $|\Omega|$ and frequency γ are in agreement with the growth rate (9) with a plus sign and the frequency of modulational perturbation, $\beta(\partial\omega/\partial k)+\delta(\partial\omega/\partial t)$, given by Eq. (6), respectively. Comparing Eqs. (31) and (33) with Eqs. (15) and (18) we see that the asymptotic solutions (31) and (33) are in complete agreement with the growing eigenfunction (15) and damping mode function (18), respectively. Therefore, we can regard the growing-and-decaying mode solution as that described by the nonlinear evolution of the unstable mode.

Next, we consider a solution consisting of a line soliton and growing-and-decaying mode, which is given by

$$f=\cosh(\Omega t+\sigma_1)+|L|e^{\xi}\cosh(\Omega t+\sigma_2)-\frac{1}{\sqrt{D}}\times(1+L_re^{\xi})\cos(\eta)+\frac{L_i}{\sqrt{D}}e^{\xi}\sin(\eta), \tag{35}$$

$$g=u_0e^{i(kx+ly-\omega t+\phi_1)}\biggl[\cosh(\Omega t+\sigma_1-i\phi_1)+|L|e^{\xi+i\phi_2}\cosh(\Omega t+\sigma_2-i\phi_1)-\frac{1}{\sqrt{D}}\times(1+L_re^{\xi+i\phi_2})\cos(\eta)+\frac{L_i}{\sqrt{D}}e^{\xi+i\phi_2}\sin(\eta)\biggr], \tag{36}$$

where

$$\xi=\kappa x+\rho y-\Gamma t+\sigma_3,$$

$$\eta=\beta x+\delta y-\gamma t+\theta,$$

$$\sin^2\frac{\phi_1}{2}=\frac{\delta^2-p\beta^2}{2ru_0^2}>0,$$

$$\sin^2\frac{\phi_2}{2}=\frac{p\kappa^2-\rho^2}{2ru_0^2}>0,$$

$$\Omega=(p\beta^2+\delta^2)\cot\frac{\phi_1}{2},$$

$$\gamma=2p\kappa\beta+2l\delta,$$

$$\Gamma=2pk\kappa+2l\rho-(p\kappa^2+\rho^2)\cot\frac{\phi_2}{2},$$

$$D=\frac{2}{1+\cos\phi_1},$$

$$L=L_r+iL_i=\frac{2ru_0^2\sin(\phi_1/2)\sin(\phi_2/2)\cos[(\phi_1-\phi_2)/2]-i(p\beta\kappa-\delta\rho)}{2ru_0^2\sin(\phi_1/2)\sin(\phi_2/2)\cos[(\phi_1+\phi_2)/2]-i(p\beta\kappa-\delta\rho)},$$

$$\sigma_1-\sigma_2=\log|L|, \tag{37}$$

and σ_3 and θ are arbitrary constants.

The solutions long before and long after growth of the growing-and-decaying mode are expressed by

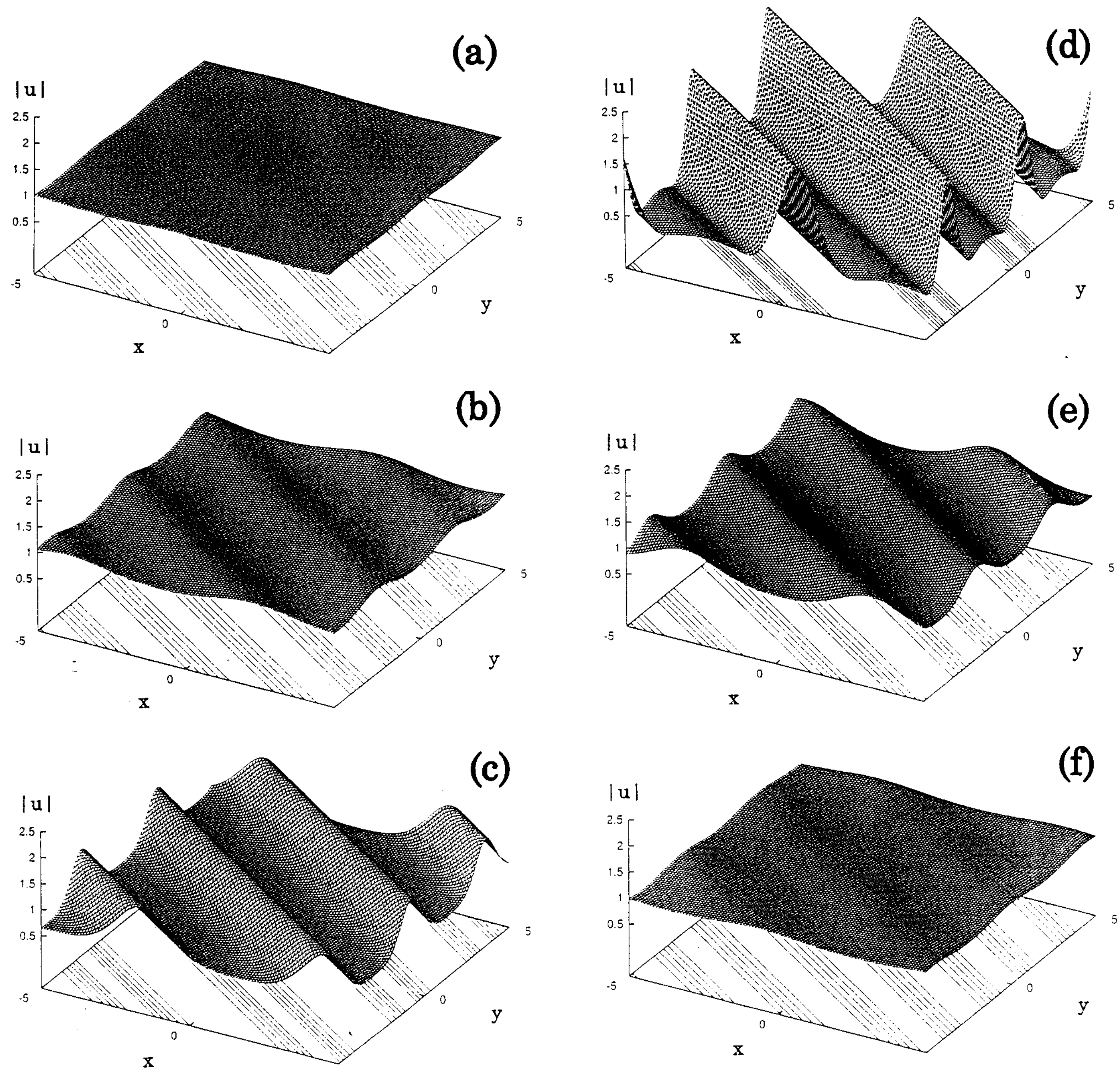

FIG. 1. The time evolution of the growing-and-decaying mode solution to the DS equation with $p=1$, $r=1$; (a) $t=-1.5$; (b) $t=-1$; (c) $t=-0.5$; (d) $t=0$; (e) $t=0.5$; (f) $t=1$. We choose $k=1$, $l=1$, $\beta=1$, and $\delta=\sqrt{2}$. The mode grows with the linear growth rate of the Benjamin-Feir instability at initial stage [(a) and (b)] takes the maximum modulation at $t=0$ Fig. [(d)] and then demodulates and returns to the unmodulated state. In this figure, x, y, and u are all dimensionless.

$$u=u_0 e^{i(kx+ly-\omega t+2\phi_1)}\frac{1+e^{\xi+2(\sigma_1-\sigma_2)+i\phi_2}}{1+e^{\xi+2(\sigma_1-\sigma_2)}},$$

$$v=-\frac{p\kappa^2}{2}\operatorname{sech}^2\frac{\xi+2(\sigma_1-\sigma_2)}{2}, \tag{38}$$

and

$$u=u_0 e^{i(kx+ly-\omega t)}\frac{1+e^{\xi+i\phi_2}}{1+e^{\xi}},$$

$$v=-\frac{p\kappa^2}{2}\operatorname{sech}^2\frac{\xi}{2}, \tag{39}$$

respectively. The effect of the growing-and-decaying mode on the line soliton is only the phase shifts $-2\phi_1$ and $2(\sigma_2-\sigma_1)$ of the plane wave and the line soliton, respectively. We can obtain the one-line soliton and N-growing-and-decaying mode solution from the solution of Satsuma and Ablowitz [17]. The solution shows that the growing and decaying mode damps to die out at sufficiently large time and only the line soliton remains finally. Therefore, these linear unstable modes do not destroy the structure of the line soliton. However, it is not known at present whether or not the infinitesimal perturbation with continuous spectrum grows into a finite amplitude and then decays into the initial state and does not destroy the structure of the line soliton.

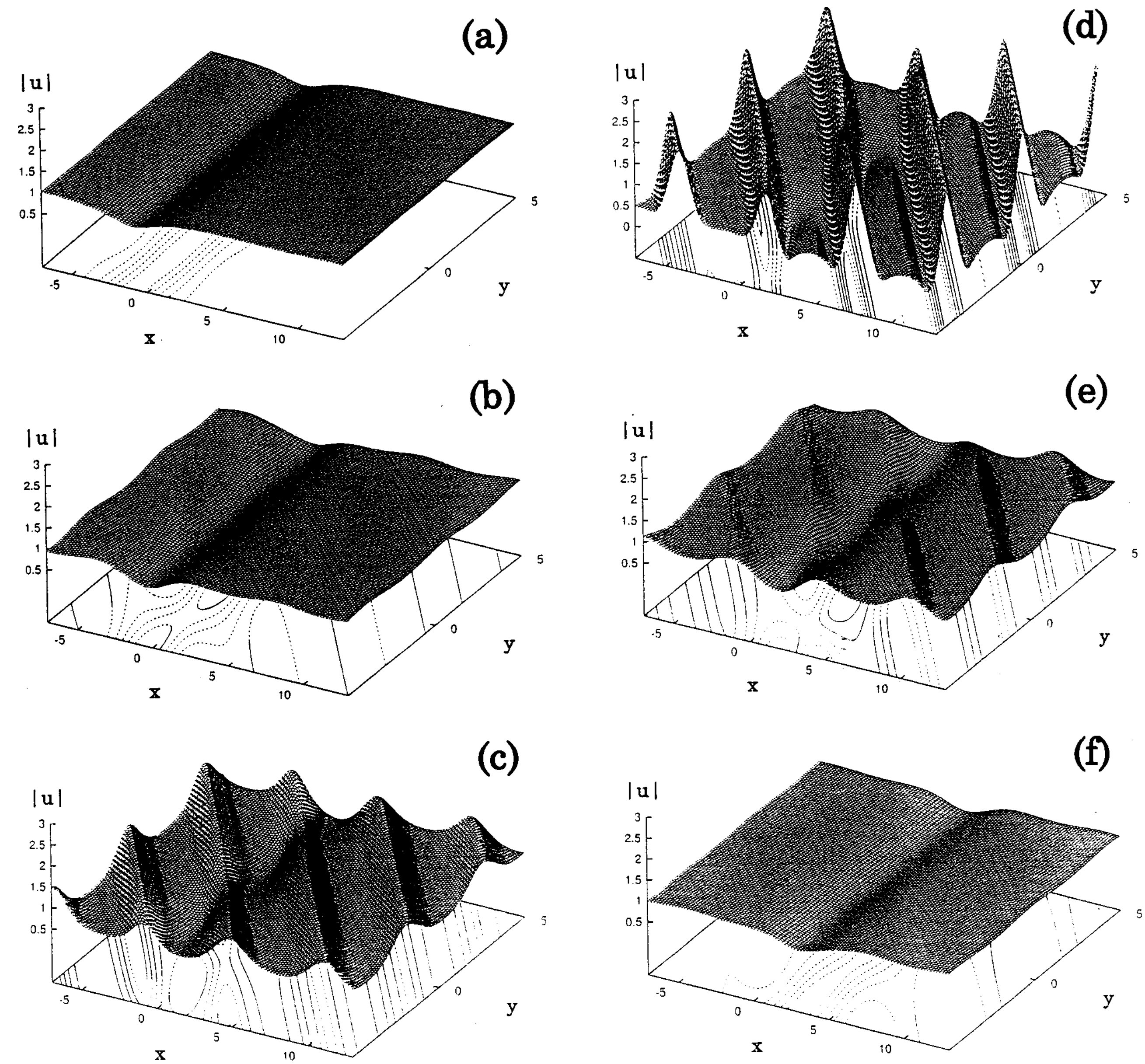

FIG. 2. The time evolution of the solution constituting of a line soliton and growing-and-decaying mode to the DS equation with $p=1$, $r=1$; (a) $t=-0.2$; (b) $t=0.4$; (c) $t=1$; (d) $t=1.6$; (e) $t=2.2$; (f) $t=2.8$. We choose $k=1$, $l=1$, $\beta=1$, $\delta=\sqrt{2}$, $\kappa=1$, and $\rho=0$. The amplitudes of the growing-and-decaying mode are very small in (a). The growing-and-decaying mode takes the maximum amplitude in (d) and almost damp to die out in (f) until only the line soliton finally remains.

III. BREATHER SOLUTION

Next, we examine an another type of the growing-and-decaying mode solution, which is a breathing localized plane pulse. The dispersion relation (23) shows that even if K_j and L_j are real, we have to take ϕ_j pure imaginary for the case $(pK_j^2-L_j^2)/2ru_0^2<0$. To obtain an analytical expression for the breathing wave solution, we set $K_1=K_2=a$, $L_1=L_2=b$ and $\phi_1=-\phi_2=i\Phi$ in Eq. (22), where a and b are real. Then, frequencies Ω_1 and Ω_2 are complex and are complex conjugate with each other and the solution is given by

$$u=u_0e^{i(kx+ly-\omega t)}\frac{\sqrt{D}\cosh\xi-\cosh\Phi\cos(\gamma t+\theta)+i\sinh\Phi\sin(\gamma t+\theta)}{\sqrt{D}\cosh\xi-\cos(\gamma t+\theta)},$$

$$v=-2pa^2D\frac{1-(1/\sqrt{D})\cosh\xi\cos(\gamma t+\theta)}{[\sqrt{D}\cosh\xi-\cos(\gamma t+\theta)]^2}, \tag{40}$$

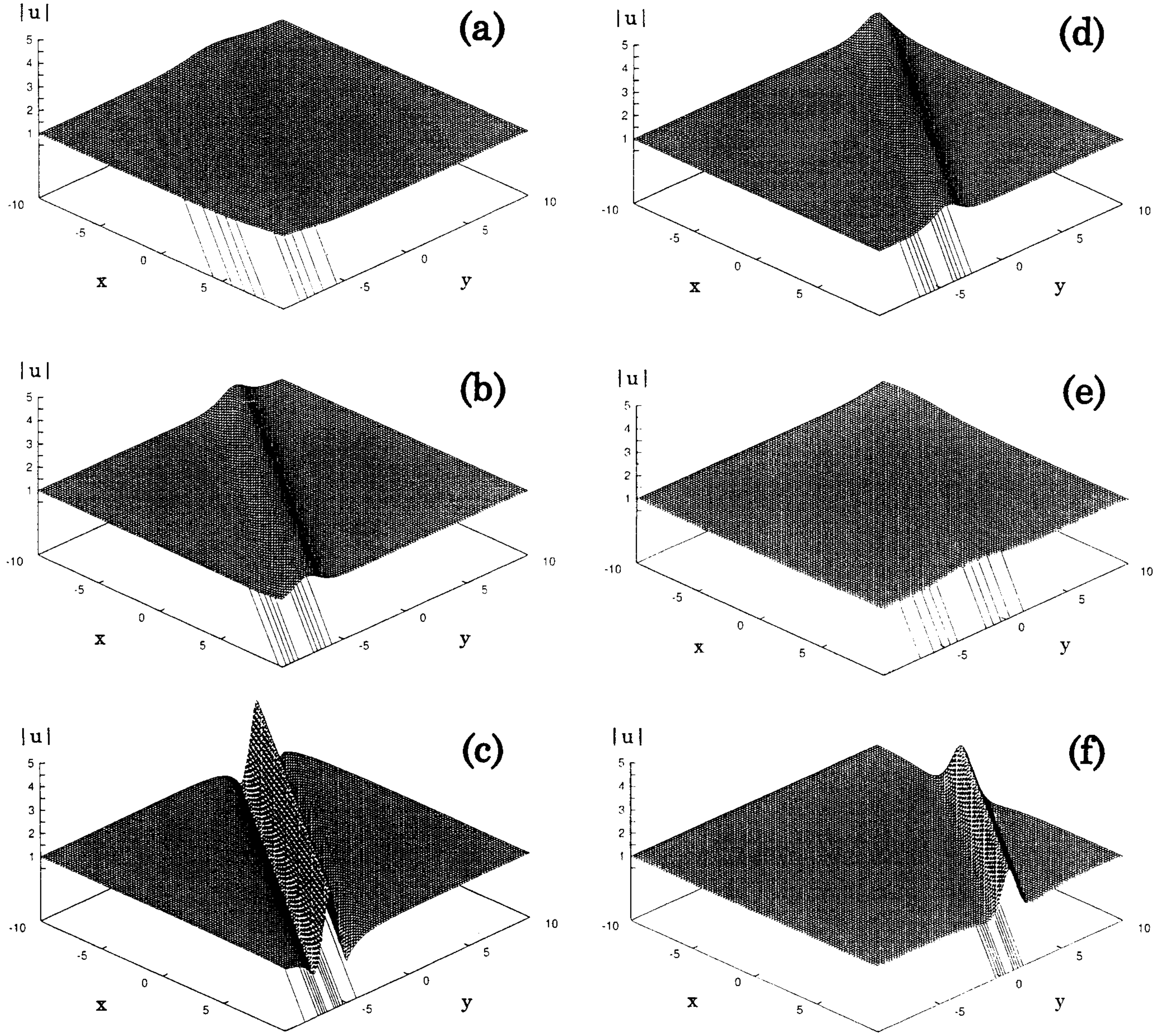

FIG. 3. The breather solution to the DS equation with $p=1$, $r=1$; (a) $t=-6$; (b) $t=-3$; (c) $t=0$; (d) $t=3$; (e) $t=6$; (f) $t=12$. We choose $k=1$, $l=1$, $a=1/2$, and $b=1/\sqrt{2}$. This solution shows that a plain wave packet propagates with breathing.

where

$$\xi=ax+by-\Omega t+\sigma,$$

$$\sinh^2\frac{\Phi}{2}=\frac{b^2-pa^2}{2ru_0^2}>0,$$

$$\Omega=2(pka+lb),$$

$$\gamma=(b^2+pa^2)\sqrt{\frac{2ru_0^2}{b^2-pa^2}+1},$$

$$D=1+\frac{b^2-pa^2}{2ru_0^2}, \tag{41}$$

where σ and θ are arbitrary phase constants. The existence condition for the nonsingular solution (40) is given by

$$\frac{b^2-pa^2}{2ru_0^2}>0, \tag{42}$$

which comes from $D>1$. A typical time evolution of this solution is shown in Fig. 3, which seems to be a breathing plane wave packet. This breather solution is an exponentially localized entity in the propagating direction and as a function of $(\gamma t+\theta)$ is periodic.

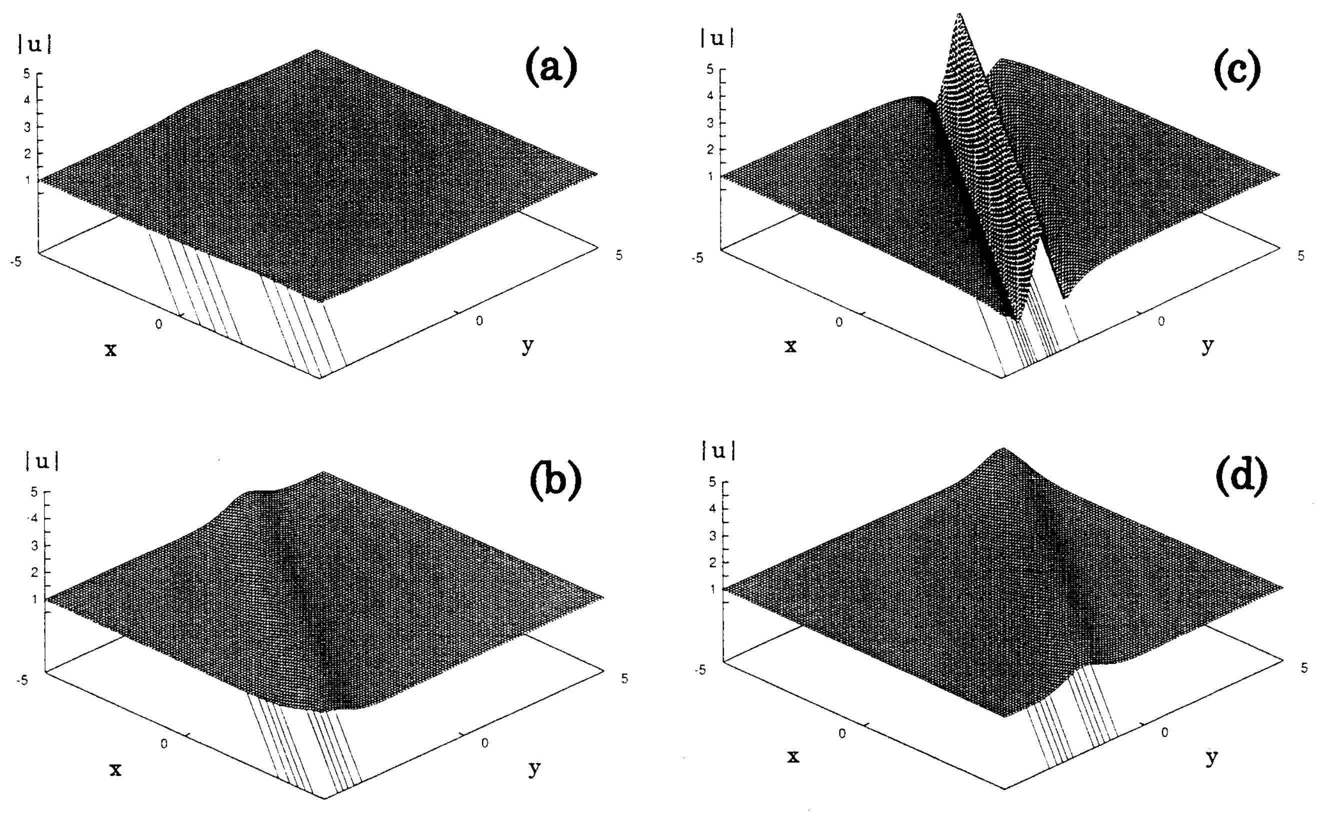

FIG. 4. The time evolution of the rational growing-and-decaying mode solution to the DS equation with $p=1$, $r=1$; (a) $t=-0.8$; (b) $t=-0.4$; (c) $t=0$; (d) $t=0.4$. We choose $k=1$, $l=1$, $c=1/2$, and $d=1/\sqrt{2}$. This solution starts from the unmodulated state. A state of maximum modulation is reached at $t=0$. After reaching maximum amplitude, the solution starts to damp and finally returns to the initial unmodulated state.

IV. RATIONAL GROWING-AND-DECAYING MODE SOLUTION

In this section, we consider the long wave limit of the growing-and-decaying mode solution (30). Putting $K_1=K_2^*=i\varepsilon c$, $L_1=L_2^*=i\varepsilon d$, $\eta_1^0=\eta_2^{0*}=\varepsilon(i\tilde{\theta}'-\tilde{\sigma}')+i\pi$, and taking the limit as $\varepsilon\to 0$, we have

$$\phi_1=\phi_2=\pm 2\varepsilon\sqrt{\frac{d^2-pc^2}{2ru_0^2}}=\varepsilon\tilde{\phi},$$

$$\Omega=\pm\varepsilon(d^2+pc^2)\sqrt{\frac{2ru_0^2}{d^2-pc^2}}=\varepsilon\tilde{\Omega},$$

$$\gamma=\varepsilon(2pkc+2ld)=\varepsilon\tilde{\gamma},$$

$$D=1+\varepsilon^2\frac{d^2-pc^2}{2ru_0^2}+O(\varepsilon^3)=1+\varepsilon^2\alpha^2+O(\varepsilon^3) \quad (43)$$

and

$$f=\varepsilon^2(\xi^2+\eta^2+\alpha^2)+O(\varepsilon^3), \quad (44)$$

$$g=u_0e^{i(kx+ly-\omega t)}[\varepsilon^2\{\xi^2+\eta^2+\alpha^2-4\alpha(\alpha\pm i\eta)\}+O(\varepsilon^3)], \quad (45)$$

where $\xi=cx+dy-\tilde{\gamma}t+\tilde{\theta}'$, $\eta=\tilde{\Omega}t+\tilde{\sigma}'$. Substitution of Eqs. (44) and (45) into Eq. (19) gives the following solution:

$$u=u_0e^{i(kx+ly-\omega t)}\left\{1-\frac{4\alpha(\alpha\pm i\eta)}{\alpha^2+\eta^2+\xi^2}\right\},$$

$$v=-4pc^2\frac{\alpha^2+\eta^2-\xi^2}{(\alpha^2+\eta^2+\xi^2)^2}. \quad (46)$$

This nonsingular solution extends in one direction and is localized in the orthogonal direction and in time as shown in Fig. 4. This solution is the rational growing and decaying mode which decays algebraically at large distance and time. It is very interesting to note that the growing-and-decaying mode solution and the breather solution can be constructed as the imbricate series of rational growing-and-decaying modes in much the same way as the 1D-NLS equation [14]. In this sense, we can regard the rational growing-and-decaying mode as the fundamental constituent of recurrent wave solutions.

V. DISCUSSION

We have shown that the DS equation has growing and decaying mode, breather, and rational growing-and-decaying mode solution. The nonlinear evolution of the monochromatic infinitesimal perturbation with a Benjamin-Feir unstable condition is described by the growing-and-decaying mode solution. The infinitesimal perturbation composed of discrete wave numbers grows a finite perturbation and then decays into the initial state, which is described by the N-growing-and-decaying mode solution. These unstable modes do not destroy the structure of the line soliton, which is described by the solution consisting of the line soliton and N-growing-and-decaying modes. It is not known whether or not these recurrent phenomena actually occur for the perturbation with a continuous spectrum. However, the existence of the rational growing-and-decaying mode solution shows the possibility that such recurrent phenomena also occur for the perturbation with a continuous spectrum. Here it should be noted that the nonlinear development of localized unstable modes on the line soliton cannot be described by the line soliton and N-growing-and-decaying mode solution. Such a stability problem on the line soliton may be described by the soliton resonant solution between the line soliton and the periodic soliton. Recently, the existence of resonant interaction between the y-periodic soliton and the line soliton was shown in the DS I equation [18]. The quasiresonant state consists of the resonant line soliton and small disturbance, if we choose parameters close to the resonant conditions. From this fact, such a stability problem on the line soliton may be treated by using the solutions of the periodic soliton resonance between the line soliton and the periodic soliton. Such investigations are in progress and will be presented elsewhere.

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